Cellularity of Chromatic Synthetic Spectra

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Abstract

We show that the ∞-category of synthetic spectra based on Morava E-theory is generated by the bigraded spheres and identify it with the ∞-category of modules over a filtered ring spectrum. The latter we show using a general method for constructing filtered deformations from t-structures on symmetric monoidal stable ∞-categories.

Pstrągowski [Pst22] defined an ∞-category Syn_R of *R-synthetic spectra* categorifying the R-Adams spectral sequence in spectra, where R is an Adams type ring spectrum. This comes with a natural notion of bigraded spheres, but unlike in spectra, it is not clear in general whether bigraded homotopy groups detect equivalences of R-synthetic spectra. If this happens, we say that Syn_R is *cellular*. The main result of this paper is to show that this holds when R is Morava E-theory. In the following, if $\mathcal{C}$ a stable ∞-category with a t-structure, we write $\mathrm{Fil}(\mathcal{C})$ for $\mathrm{Fun}(\mathbf{Z}^{\mathrm{op}}, \mathcal{C})$, and we let $\mathrm{Wh}\colon \mathcal{C} \to \mathrm{Fil}(\mathcal{C})$ denote its Whitehead filtration functor.

Theorem A. *Let E denote a Morava E-theory at an arbitrary prime and height.*

(1) *The ∞-category Syn_E is cellular.*

(2) *There is a symmetric monoidal equivalence*

$$\mathrm{Syn}_E \xrightarrow{\simeq} \mathrm{Mod}_{\mathrm{map}(\nu\mathbf{S},\, \mathrm{Wh}(\tau^{-1}\nu\mathbf{S}))}(\mathrm{FilSp})$$

such that νX is sent to $\mathrm{Tot}(\mathrm{Wh}(E^{\otimes[\bullet]} \otimes X))$ whenever $X \in \mathrm{Sp}$ is E-nilpotent complete.

Proof. See Theorem 1.4 and Corollary 2.5 below. ∎

Pstrągowski showed in [Pst22] that MU-synthetic spectra are cellular. Later, Lawson [Law24] generalised this, showing that Syn_R is cellular whenever R is connective. This does not imply Theorem A (except at height 0): indeed, since E is $\mathbf{F}_p$-acyclic for all p, the ∞-category Syn_E is not equivalent to Syn_R for any connective R.

As explained by Burklund–Hahn–Senger [BHS22, Appendix C], Lawson [Law24, Corollary 6.1], and Pstrągowski [Pst24, Sections 3.3 and 3.4], if Syn_R is cellular, then this leads to a *filtered model* for R-synthetic spectra. This is the second part of Theorem A. Although these types of results are thus well-known, we include a proof in order to highlight the role that t-structures play in obtaining such a result. Namely, the main hurdle in proving such statements is constructing a FilSp-module structure on Syn_R, or equivalently an action of the monoidal poset $\mathbf{Z}$. Unlike the discrete monoid $\mathbf{Z}^{\mathrm{disc}}$, the monoidal poset $\mathbf{Z}$ has no simple universal property. We include a short discussion of how t-structures give rise to FilSp-module structures; this is implicit in the previously cited works.

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1 Cellularity

Fix a homotopy-associative ring spectrum R of Adams type. We follow the terminology and notation of [Pst22]. In addition, it will be useful to introduce the following concept.

Definition 1.1. A cofibre sequence $X \to Y \to Z$ of spectra is called ***R*-exact** if it yields a short exact sequence on R_*-homology:

$$0 \longrightarrow R_*X \longrightarrow R_*Y \longrightarrow R_*Z \longrightarrow 0.$$

Definition 1.2. A (not necessarily stable) full subcategory of Sp is called ***R*-thick** if it is closed under

(1) arbitrary suspensions,

(2) retracts,

(3) 2-out-of-3 for R-exact cofibre sequences.

For a collection of spectra $J \subseteq \mathrm{Sp}$, we denote by $\mathrm{Thick}^R(J)$ the smallest R-thick subcategory of Sp containing J.

It is straightforward to see that the subcategory $\mathrm{Sp}_R^{\mathrm{fp}}$ of finite R-projective spectra is R-thick. It follows that

$$\mathrm{Thick}^R(\mathbf{S}) \subseteq \mathrm{Sp}_R^{\mathrm{fp}}.$$

Proposition 1.3. *If we have an equality*

$$\mathrm{Thick}^R(\mathbf{S}) = \mathrm{Sp}_R^{\mathrm{fp}},$$

then Syn_R *is cellular.*

We do not know if this is a necessary condition for Syn_R to be cellular.

Proof. The ∞-category Syn_R is generated under colimits by $\mathrm{Thick}(\nu P \mid P \in \mathrm{Sp}_R^{\mathrm{fp}})$; see [Pst22, Remark 4.14]. It therefore suffices to show that this thick subcategory is contained in $\mathrm{Thick}(\nu \mathbf{S}^n \mid n \in \mathbf{Z})$. The assumption $\mathrm{Sp}_R^{\mathrm{fp}} \subseteq \mathrm{Thick}^R(\mathbf{S})$ implies that

$$\nu(\mathrm{Sp}_R^{\mathrm{fp}}) \subseteq \nu\,\mathrm{Thick}^R(\mathbf{S}).$$

Recall that ν is additive and sends R-exact cofibre sequences to cofibre sequences; see [Pst22, Lemma 4.23]. Using this and the fact that Sp is idempotent-complete, it follows from the definition of an R-thick subcategory that we have the containment $\nu\,\mathrm{Thick}^R(\mathbf{S}) \subseteq \mathrm{Thick}(\nu \mathbf{S}^n \mid n \in \mathbf{Z})$. This finishes the proof. ∎

Theorem 1.4. *Let p be a prime and let $h \geqslant 0$ be an integer, and let E denote a Morava E-theory at height h. Then Syn_E is cellular.*

In the proof, let K denote a Morava K-theory corresponding to E.

Lemma 1.5. *Let $X \to Y$ be a map of finite E-projective spectra. Then $E_*X \to E_*Y$ is surjective if and only if $K_*X \to K_*Y$ is surjective.*

Proof. Before we prove this, let us recall that since E_* is a graded power series ring over a local ring, every graded projective module over it is free. It follows that for every $P \in \mathrm{Sp}_E^{\mathrm{fp}}$, we have a natural isomorphism

$$K_* \otimes_{E_*} E_*P = \pi_*(K) \otimes_{\pi_*E} \pi_*(E \otimes P) \xrightarrow{\cong} \pi_*(K \otimes_E (E \otimes P)) = \pi_*(K \otimes P) = K_*P.$$

We proceed to the proof.

($\Longrightarrow$) Suppose that $E_*X \to E_*Y$ is surjective. Tensoring this with K_* yields a surjective map, which by the above isomorphism can be identified with the map $K_*X \to K_*Y$.

($\Longleftarrow$) Suppose that $K_*X \to K_*Y$ is surjective, and write $C_* = \mathrm{coker}(E_*X \to E_*Y)$. The isomorphism above and right exactness of the tensor product imply that $\mathrm{coker}(K_*X \to K_*Y) \cong K_* \otimes_{E_*} C_* = 0$. Since C_* is finitely generated, Nakayama's lemma applies, which implies that $C_* = 0$, so that $E_*X \to E_*Y$ is surjective. ∎

Proof of Theorem 1.4. By Proposition 1.3, it suffices to show that every finite E-projective spectrum P is in $\mathrm{Thick}^E(\mathbf{S})$. By suspending sufficiently many times, we may without loss of generality assume that P is connective. We will now proceed by ascending induction on $\mathbf{N} \times \mathbf{N}$ with the lexicographical ordering, where to every finite E-projective spectrum P we assign

$$(\dim P,\ \mathrm{rk}\, P) \in \mathbf{N} \times \mathbf{N},$$

where $\dim P$ denotes the dimension of the top cells of P, and where $\operatorname{rk} P := \operatorname{rk}_{E_*}(E_*P)$. For the base case, where $\dim P = 0$, it follows from connectivity of P that P is a sum of zero-spheres, which is obviously in $\mathrm{Thick}^E(\mathbf{S})$. We proceed to the inductive step.

As a first case, assume that there exists a top-dimensional cell (of dimension $d := \dim P$) for which the projection onto the top cell $P \to \mathbf{S}^d$ induces a surjection on E_*-homology. Let F denote the fibre of this map. A surjection of projective modules is always split, so $E_*F = \ker(E_*P \to E_*\mathbf{S}^d)$ is projective, and therefore $F \in \mathrm{Sp}_E^{\mathrm{fp}}$. Moreover, we have $\operatorname{rk} F = \operatorname{rk} P - 1$ and $\dim F \leqslant \dim P$, so by our induction hypothesis, we have $F \in \mathrm{Thick}^E(\mathbf{S})$. Since $F \to P \to \mathbf{S}^d$ is an E-exact fibre sequence, it follows that $P \in \mathrm{Thick}^E(\mathbf{S})$.

We may therefore assume that for every choice of top cell for P, the projection onto the top cell is not surjective on E_*. A choice of cellular filtration on P gives us a cofibre sequence

$$\mathrm{Sk}_{d-1}(P) \longrightarrow P \longrightarrow \bigoplus_{\mathrm{Cell}_d(P)} \mathbf{S}^d,$$

where $\mathrm{Cell}_d(P)$ is the set of d-dimensional cells for the chosen cellular filtration. Consider now the map on K-theory induced by projection onto a top-dimensional cell

$$K_*P \longrightarrow K_*\mathbf{S}^d \cong K_{*-d}.$$

As K_* is a (graded) field and the right hand side is of rank 1, this map is either surjective or zero. If it were surjective, then by Lemma 1.5 it would have also been surjective on E_*, in contradiction to our assumption. It must therefore be the zero map on K_*. As this argument applies to all top cells, we conclude that $K_*(P) \to K_*(\bigoplus_{\mathrm{Cell}_d(P)} \mathbf{S}^d)$ is also the zero map. It follows that $\mathrm{Sk}_{d-1}(P) \to P$ is surjective on K_*, so by Lemma 1.5 it is also surjective on E_*. We learn that the cofibre sequence

$$\bigoplus_{\mathrm{Cell}_d(P)} \mathbf{S}^{d-1} \longrightarrow \mathrm{Sk}_{d-1}(P) \longrightarrow P$$

is E_*-exact. But $\dim(\mathrm{Sk}_{d-1}(P)) = \dim(P) - 1$, so by the induction hypothesis, we have $\mathrm{Sk}_{d-1}(P) \in \mathrm{Thick}^E(\mathbf{S})$. We conclude that $P \in \mathrm{Thick}^E(\mathbf{S})$, and we are done. ■

2 Deformations from t-structures

The ∞-category of **filtered spectra** is defined as $\mathrm{FilSp} = \mathrm{Fun}(\mathbf{Z}^{\mathrm{op}}, \mathrm{Sp})$, where we consider $\mathbf{Z}$ as a poset under the usual ordering. We regard this as a symmetric monoidal ∞-category under Day convolution (where $\mathbf{Z}$ carries addition); note that this turns it into a presentably symmetric monoidal ∞-category, i.e., an $\mathbf{E}_\infty$-algebra in Pr^{L}. This category comes with a notion of shifting: if X is a filtered spectrum, then we write $X(n)$ for the filtered spectrum given by

$$\mathbf{Z}^{\mathrm{op}} \xrightarrow{-n} \mathbf{Z}^{\mathrm{op}} \xrightarrow{X} \mathrm{Sp}.$$

Note that this functor is equivalently given by tensoring with $\mathbf{1}(n)$, where $\mathbf{1}$ denotes the unit of FilSp. The connecting maps of X induce natural transformations $X(n+1) \to X(n)$ for every n.

Following [Bar23], a **deformation** is a module over FilSp in Pr^{L}. If $\mathcal{C}$ is a deformation and $X \in \mathcal{C}$, then we can define $X(n)$ as $\mathbf{1}(n) \otimes X$. This gives rise to a filtered mapping spectrum: for $X, Y \in \mathcal{C}$, we define $\mathrm{filmap}_{\mathcal{C}}(X, Y)$ to be the filtered spectrum given by

$$\mathrm{map}_{\mathcal{C}}(\mathbf{1}(-) \otimes X,\, Y),$$

where $\mathrm{map}_{\mathcal{C}}$ denotes the mapping spectrum functor of the stable ∞-category $\mathcal{C}$.

Theorem 2.1 (Filtered Schwede–Shipley; Pstrągowski [Pst24], Proposition 3.16). *Let $\mathcal{C}$ be a deformation, and let $X \in \mathcal{C}$. Then the following are equivalent.*

(a) *The functor* $\mathrm{filmap}_{\mathcal{C}}(X, -)\colon \mathcal{C} \to \mathrm{Mod}_{\mathrm{filmap}_{\mathcal{C}}(X,X)}(\mathrm{FilSp})$ *is a symmetric monoidal equivalence.*

(b) *The object X is compact, and the objects $X(n)$ for $n \in \mathbf{Z}$ generate $\mathcal{C}$ under (de)suspensions and colimits.*

Applying this result requires obtaining a FilSp-module structure on $\mathcal{C}$. As noted in the introduction, obtaining this structure can be difficult, because the poset $\mathbf{Z}$ is not free as a symmetric monoidal ∞-category. An important source of such a structure is a monoidal t-structure on $\mathcal{C}$, as we now explain.

Suppose $\mathcal{C}$ is a symmetric monoidal stable ∞-category with a compatible t-structure. Recall that the Whitehead filtration functor $\mathrm{Wh}\colon \mathcal{C} \to \mathrm{Fil}(\mathcal{C})$ is lax symmetric monoidal, being the composite

$$\mathcal{C} \xrightarrow{\mathrm{Const}} \mathrm{Fil}(\mathcal{C}) \xrightarrow{\tau_{\geqslant 0}^{\mathrm{diag}}} \mathrm{Fil}(\mathcal{C})$$

where the first functor is the constant-filtered-object functor, and the second is the connective cover with respect to the diagonal t-structure (see, e.g., [Hed20, Proposition II.1.23]), both of which are canonically lax symmetric monoidal. If A is an $\mathbf{E}_\infty$-algebra in $\mathcal{C}$, we therefore obtain an $\mathbf{E}_\infty$-algebra $\mathrm{Wh}\, A$ in $\mathrm{Fil}(\mathcal{C})$, which by the equivalence $\mathrm{CAlg}(\mathrm{Fil}(\mathcal{C})) \simeq \mathrm{Fun}^{\mathrm{lax}}(\mathbf{Z}^{\mathrm{op}}, \mathcal{C})$ of [HA, Example 2.2.6.9] is the same as a lax symmetric monoidal structure on the functor $\mathrm{Wh}\, A\colon \mathbf{Z}^{\mathrm{op}} \to \mathcal{C}$.

Definition 2.2. Let $\mathcal{C}$ be a symmetric monoidal ∞-category with a compatible t-structure. Let A be an $\mathbf{E}_\infty$-algebra in $\mathcal{C}$. We say that A is **t-strict** if the lax symmetric monoidal functor $\mathrm{Wh}\, A\colon \mathbf{Z}^{\mathrm{op}} \to \mathcal{C}$ is strong symmetric monoidal.

Lemma 2.3. *Let $\mathcal{C}$ and A be as in Definition* 2.2*. Then A is t-strict if and only if*

(a) *The map $\mathbf{1} \to \tau_{\geqslant 0} A$ induced by the unit of A is an isomorphism.*

(b) *The natural map $\tau_{\geqslant n} A \otimes \tau_{\geqslant m} A \to \tau_{\geqslant n+m} A$ is an isomorphism for every $n, m \in \mathbf{Z}$.*

Proof. Condition (a) says that Wh A preserves empty products, while condition (b) says it preserves binary products. ■

Theorem 2.4. *Let $\mathcal{C}$ be a presentably symmetric monoidal stable ∞-category equipped with a compatible t-structure. Let A be an $\mathbf{E}_\infty$-algebra in $\mathcal{C}$. Suppose that*

(a) *A is t-strict,*

(b) *the unit $\mathbf{1}$ of $\mathcal{C}$ is compact,*

(c) *the objects $\Sigma^n \tau_{\geqslant m} A$ for $n, m \in \mathbf{Z}$ generate $\mathcal{C}$ under colimits.*

Then the functor

$$\operatorname{map}_{\mathcal{C}}(\mathbf{1}, \operatorname{Wh} A \otimes -) \colon \mathcal{C} \xrightarrow{\simeq} \operatorname{Mod}_{\operatorname{map}_{\mathcal{C}}(\mathbf{1}, \operatorname{Wh} A)}(\mathrm{FilSp})$$

is an equivalence of symmetric monoidal ∞-categories.

Proof. The symmetric monoidal functor

$$\mathbf{Z} \xrightarrow{n \mapsto -n} \mathbf{Z}^{\mathrm{op}} \xrightarrow{\operatorname{Wh} A} \mathcal{C}, \qquad n \longmapsto \tau_{\geqslant -n} A$$

induces a symmetric monoidal left adjoint $\mathrm{FilSp} \to \mathcal{C}$, giving $\mathcal{C}$ the structure of a symmetric monoidal deformation. Applying Theorem 2.1, all that remains is to identify $\operatorname{filmap}_{\mathcal{C}}(\mathbf{1}, -)$ with $\operatorname{map}(\mathbf{1}, \operatorname{Wh} A \otimes -)$. By assumption of t-strictness, we see that for every $n \in \mathbf{Z}$, we have a natural (in n) isomorphism

$$\operatorname{map}_{\mathcal{C}}(\mathbf{1}, \tau_{\geqslant n} A) \cong \operatorname{map}_{\mathcal{C}}(\tau_{\geqslant -n} A, \mathbf{1}).$$

The right-hand side is (by definition of the deformation structure) the value at filtration n of the filtered spectrum $\operatorname{filmap}_{\mathcal{C}}(\mathbf{1}, \mathbf{1})$, proving the claim. ■

Corollary 2.5. *Let E denote a Morava E-theory. Then there is a symmetric monoidal equivalence*

$$\mathrm{Syn}_E \xrightarrow{\simeq} \operatorname{Mod}_{\operatorname{map}(\nu\mathbf{S}, \operatorname{Wh}(\tau^{-1}\nu\mathbf{S}))}(\mathrm{FilSp})$$

such that νX is sent to $\operatorname{Tot}(\tau_{\geqslant \star}(E^{\otimes[\bullet]} \otimes X))$ whenever X is a E-nilpotent complete spectrum.

Proof of Corollary 2.5. In the case $\mathcal{C} = \mathrm{Syn}_R$, the τ-inverted unit $\tau^{-1}\nu\mathbf{S}$ is a t-strict $\mathbf{E}_\infty$-algebra. The resulting functor $\operatorname{map}(\mathbf{1}, \operatorname{Wh} A)$ is called the *signature functor* in [CD24; CDvN24a; CDvN24b], where it is denoted by σ; in [BHS22, Appendix C], this functor is denoted by i_*. The result now follows by using [CDvN24a, Proposition 1.25]. ■

Remark 2.6. For general R, the underlying filtered spectrum $\sigma\nu(\mathbf{S})$ is, after completion, equivalent to the décalage of the cosimplicial Adams resolution:

$$\operatorname{Tot}(\tau_{\geqslant \star}(R^{\otimes[\bullet]})).$$

If R is an $\mathbf{E}_\infty$-ring, then this equivalence is naturally one of filtered $\mathbf{E}_\infty$-rings; see [CDvN24a, Proposition 1.25]. If the sphere is R-nilpotent complete, then this is even true without completion of filtered spectra. Alternatively, as in [BHS22, Proposition C.22], if R is an $\mathbf{E}_\infty$-ring, the above implies that σ induces a symmetric monoidal equivalence

$$\mathrm{Mod}_{\sigma(\nu\mathbf{S})^\wedge_\tau}(\mathrm{Syn}_R) \xrightarrow{\simeq} \mathrm{Mod}_{\mathrm{Tot}(\tau_{\geqslant\star}(R^{[\bullet]}))}(\mathrm{FilSp}).$$

References

[Bar23] S. Barkan. *Chromatic Homotopy Is Monoidally Algebraic at Large Primes*. 5th June 2023. arXiv: 2304.14457v2 [math]. Preprint.

[BHS22] R. Burklund, J. Hahn and A. Senger. *Galois reconstruction of Artin-Tate $\mathbb{R}$-motivic spectra*. 30th June 2022. arXiv: 2010.10325v2 [math]. Preprint.

[CD24] C. Carrick and J. M. Davies. *A synthetic approach to detecting v_1-periodic families*. 29th Jan. 2024. arXiv: 2401.16508 [math]. Preprint.

[CDvN24a] C. Carrick, J. M. Davies and S. van Nigtevecht. *Descent Spectral Sequences through Synthetic Spectra*. 1st July 2024. arXiv: 2407.01507v1 [math]. Preprint.

[CDvN24b] C. Carrick, J. M. Davies and S. van Nigtevecht. *The Descent Spectral Sequence for Topological Modular Forms*. 2nd Dec. 2024. arXiv: 2412.01640v1 [math]. Preprint.

[Hed20] A. P. Hedenlund. 'Multiplicative Tate Spectral Sequences'. PhD thesis. University of Oslo, 2020. URL: https://www.mn.uio.no/math/personer/vit/rognes/theses/hedenlund-thesis.pdf.

[Law24] T. Lawson. *Synthetic Spectra Are (Usually) Cellular*. 6th Feb. 2024. arXiv: 2402.03257 [math]. Preprint.

[HA] J. Lurie. 'Higher Algebra'. 18th Sept. 2017. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf.

[Pst22] P. Pstrągowski. 'Synthetic Spectra and the Cellular Motivic Category'. *Inventiones mathematicae* (20th Dec. 2022). DOI: 10.1007/s00222-022-01173-2.

[Pst24] P. Pstrągowski. *Perfect Even Modules and the Even Filtration*. 24th Oct. 2024. arXiv: 2304.04685v2 [math]. Preprint.