Profinite descent for Picard groups

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These are some slightly expanded notes of a talk I gave in spring 2024 edition of the *Freudenthal topology seminar* in Utrecht. My goal was to give an introduction to the approach to computing $\mathrm{K}(n)$-local Picard groups that Itamar Mor presents in his thesis [Mor23a; Mor23b]. His thesis contains much more, like a discussion of Brauer groups for instance, but I restricted myself to the Picard case. Moreover, I did not present any in-depth proofs, but rather wanted to focus on the overall ideas and strategies. This is particularly evident in §5 of these notes, where I merely give the statements of Mor's results; this does not do justice to the amount of work involved in setting up this machinery.

This talk features several spectral sequences. I have not included pictures of these spectral sequences in these notes (simply out of laziness), but I have included references to places to find these pictures. I highly recommend the readers to look at these: I have found that any story involving spectral sequences come to life much more when looking at a picture in conjunction with the formulas.

I would like to thank the audience for their active participation. In particular, I thank Gijs Heuts and Tobias Lenz for pointing out some small mistakes and for their helpful comments during the talk. Any mistakes in these written notes however remain my own. If you find any typos or mistakes (or are otherwise confused by a comment I make), please do not hesitate to let me know.

1 Introduction

Definition 1.1. Let $\mathcal{C}$ be a monoidal ∞-category. The **Picard group** of $\mathcal{C}$ is the group given by

$$\mathrm{Pic}(\mathcal{C}) := \{ \otimes\text{-invertible objects of } \mathcal{C} \}/\simeq.$$

If $\mathcal{C}$ is symmetric monoidal, then the group $\mathrm{Pic}(\mathcal{C})$ is an abelian group. If A is an $\mathbf{E}_\infty$-ring, then we will abbreviate

$$\mathrm{Pic}(A) := \mathrm{Pic}(\mathrm{Mod}_A).$$

There is a better version of the Picard group where we remember the equivalences between these objects, instead of identifying equivalent objects. As we will see momentarily, even if one is only interested in computing $\mathrm{Pic}(\mathcal{C})$, it is nonetheless a good idea to use this more structured object instead.

Variant 1.2. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. The **Picard space** of $\mathcal{C}$ is the full subgroupoid $\mathfrak{Pic}(\mathcal{C})$ of $\mathcal{C}^{\simeq}$ on the objects in $\mathrm{Pic}(\mathcal{C})$. The symmetric monoidal structure on $\mathcal{C}$ equips $\mathfrak{Pic}(\mathcal{C})$ with the structure of an $\mathbf{E}_\infty$-monoid in spaces. As it is a grouplike $\mathbf{E}_\infty$-space, it corresponds uniquely to a connective spectrum $\mathfrak{pic}(\mathcal{C})$ called the **Picard spectrum** of $\mathcal{C}$.

Remark 1.3. At the moment I am ignoring size issues; see Proposition 2.7 for a treatment of these.

From the perspective of stable homotopy theory, the Picard group can be motivated by saying it is the largest collection of objects on which one can index the homotopy groups in $\mathcal{C}$.

Example 1.4. We have

$$\mathrm{Pic}(\mathrm{Sp}) = \{ \mathbf{S}^n \mid n \in \mathbf{Z} \} \cong \mathbf{Z},$$

reflecting why we index homotopy groups of spectra by the integers. ▲

Example 1.5. We have an equality

$$\mathrm{Pic}(\mathrm{Mod}_{\mathrm{KU}}) = \{ \Sigma^n \mathrm{KU} \mid n \in Z \}.$$

Indeed, every such shift is certainly an invertible KU-module. Since the homotopy groups $\pi_* \mathrm{KU} = \mathbf{Z}[u^{\pm}]$ form a very simple ring*, it is not too difficult to show that these are indeed all invertible objects; see [Wol98]. However, writing it in this manner is very redundant: as groups, we have

$$\mathrm{Pic}(\mathrm{Mod}_{\mathrm{KU}}) \cong \mathbf{Z}/2,$$

being generated by $\Sigma\mathrm{KU}$. This is a very convoluted way of saying that homotopy groups of KU-modules are 2-periodic. ▲

The main goal of today is to understand

$$\mathrm{Pic}(\mathrm{Sp}_{\mathrm{K}(n)}).$$

This was first considered by Hopkins, Mahowald and Sadofsky [HMS94], who, among other things, compute this at height $n = 1$ (at all primes). The goal of today is to follow Mor's recent approach [Mor23a], which, among other things, reproduces the height $n = 1$ computation.

*Specifically, what we use is that it has cohomological dimension 1, because it is a PID.

The idea is to use *Galois descent* for Picard groups, but to do this in a way that carefully keeps track of the profinite nature of Morava E-theory. To give a more gentle introduction to these ideas, I want to start by discussing an easier example, namely, the case of real K-theory KO.

2 The Picard group of KO

Our warm-up goal will be to compute Pic(KO). Note that all shifts of KO are in there, and since KO is 8-periodic, we learn that this at least contains $\mathbf{Z}/8$ as a subgroup. We cannot however proceed in the same way as in Example 1.5, because $\pi_*\mathrm{KO}$ is not a very nice ring algebraically speaking. What we can do is start with the more elementary case of Pic(KU), and use this to compute Pic(KO). The key input is the identification

$$\mathrm{KO} \simeq \mathrm{KU}^{\mathrm{h}C_2}. \tag{2.1}$$

The idea then is to compute Pic(KO) by taking the homotopy fixed points of Pic(KU).

Example 2.2. As a further warm-up, we review how to compute $\pi_*\mathrm{KO}$. The sentiment is the same: we regard the case of $\pi_*\mathrm{KU}$ as easy, and try to use a fixed point method to deduce the case of KO from this.

Concretely, there is a spectral sequence taking us from the algebraic notion of fixed points to the homotopy fixed points. This is the *homotopy fixed point spectral sequence* (HFPSS), which is of the form†

$$\mathrm{E}_2^{k,s} \cong \mathrm{H}^s(C_2, \pi_{k+s}\mathrm{KU}) \implies \pi_k(\mathrm{KU}^{\mathrm{h}C_2}).$$

The identification from (2.1) tells us that this in fact abuts to $\pi_*\mathrm{KO}$. The spectral sequence moreover converges strongly, allowing us to truly compute $\pi_*\mathrm{KO}$. A picture of this is reproduced in, e.g., [MS16, §7, Figure 3]. ▲

We would like to apply this to the case of Picard groups. The ∞-category of KU-modules inherts a C_2-action from KU, so we can talk about the fixed points

$$(\mathrm{Mod}_{\mathrm{KU}})^{\mathrm{h}C_2}$$

as an ∞-category. This is in fact the correct approach, as a special case of *Galois descent*.

Theorem 2.3 (Galois descent, [MS16], Theorem 3.3.1). *Let $E \to F$ be a faithful Galois extension of $\mathbf{E}_\infty$-rings with finite Galois group G. Then we have an equivalence of symmetric monoidal ∞-categories*

$$\mathrm{Mod}_E \simeq (\mathrm{Mod}_F)^{\mathrm{h}G}.$$

†Throughout this talk, I am using Adams grading when writing spectral sequences. Usually people depict spectral sequences in this way in pictures, while using a different grading in formulas; I find this unhelpful.

Remark 2.4. Compare this to the algebraic situation of the Galois extension $\mathbf{R} \to \mathbf{C}$. The statement $\mathrm{Mod}_{\mathbf{R}} \simeq (\mathrm{Mod}_{\mathbf{C}})^{hC_2}$ boils down to the statement that an $\mathbf{R}$-vector space is equivalent to a $\mathbf{C}$-vector space together with a conjugate-linear endomorphism squaring to the identity (a.k.a. a conjugation action).

We cannot simply apply Pic to this equivalence however, as the functor Pic does not respect limits. This is why we introduced the more homotopical version $\mathfrak{pic}$ in Variant 1.2 earlier.

Theorem 2.5 ([MS16], Proposition 2.2.3). *The functor*

$$\mathfrak{pic}\colon \mathrm{CAlg}(\mathrm{Cat}_\infty) \longrightarrow \mathrm{Sp}_{\geqslant 0}$$

preserves small limits.

Remark 2.6. I write $\mathrm{CAlg}(\mathrm{Cat}_\infty)$ for the ∞-category of (small) symmetric monoidal ∞-categories. The forgetful functor $\mathrm{CAlg}(\mathrm{Cat}_\infty) \to \mathrm{Cat}_\infty$ preserves and detects limits, so the underlying ∞-category of the symmetric monoidal limit is computed as a limit of ∞-categories. Limits in $\mathrm{Sp}_{\geqslant 0}$ are computed as connective covers of limits in Sp, using that $\tau_{\geqslant 0}\colon \mathrm{Sp} \to \mathrm{Sp}_{\geqslant 0}$ is right adjoint to the inclusion (as with any t-structure; see [HA, §1.2.1]).

One should be a little careful in applying the above result about $\mathfrak{pic}$, because so far, I have been applying $\mathfrak{pic}$ to large ∞-categories, rather than small ones. I will include the details now, but as this is a more technical point, the reader can safely skip this on first reading.

Proposition 2.7. *The functor*

$$\mathfrak{pic}\colon \mathrm{Pr}^{\mathrm{L}} \longrightarrow \widehat{\mathrm{Sp}}_{\geqslant 0}$$

lands in small spectra, and the resulting functor

$$\mathfrak{pic}\colon \mathrm{Pr}^{\mathrm{L}} \longrightarrow \mathrm{Sp}_{\geqslant 0}$$

preserves small limits.

Proof. The inclusion[‡]

$$\mathrm{Pr}^{\mathrm{L}} \subseteq \widehat{\mathrm{Cat}}_\infty$$

preserves small limits by [HTT, Proposition 5.5.3.13], where the target denotes the (very large) ∞-category of potentially large ∞-categories. Interpreting Theorem 2.5 one universe higher, we obtain a functor

$$\mathfrak{pic}\colon \widehat{\mathrm{Cat}}_\infty \longrightarrow \widehat{\mathrm{Sp}}_{\geqslant 0}, \quad \mathcal{C} \longmapsto \mathfrak{pic}(\mathcal{C})$$

to potentially large connective spectra, and this functor preserves large limits. Precomposing this with the inclusion of Pr^{L}, we obtain a functor

$$\mathfrak{pic}\colon \mathrm{Pr}^{\mathrm{L}} \longrightarrow \widehat{\mathrm{Sp}}_{\geqslant 0}, \quad \mathcal{C} \longmapsto \mathfrak{pic}(\mathcal{C})$$

[‡]Beware that this is not a full subcategory.

that preserves small limits by the previous comment. By [MS16, Remark 2.1.4], this lands in small spectra. Lastly, the inclusion of small spectra into potentially large spectra also preserves small limits, finishing the argument. ■

Corollary 2.8. *We have an equivalence*

$$\mathfrak{pic}(\mathrm{KO}) \simeq \tau_{\geqslant 0}\, \mathfrak{pic}(\mathrm{KU}).$$

Proof. This follows from Galois descent, using that limits in $\mathrm{Sp}_{\geqslant 0}$ are computed as connective covers of limits in Sp (see Remark 2.6). ■

This is what we need to start understanding Pic(KO).

Proposition 2.9. *The homotopy fixed point spectral sequence for the C_2-action on $\mathfrak{pic}(\mathrm{KU})$ takes the form*

$$\mathrm{E}_2^{k,s} \cong \mathrm{H}^s(C_2,\, \pi_{k+s}\mathfrak{pic}(\mathrm{KU})) \implies \pi_k\mathfrak{pic}(\mathrm{KO})$$

provided $k \geqslant 0$, and is strongly convergent in that range.

Remark 2.10. The spectral sequence does not converge to the indicated abutment for $k < 0$. These negative homotopy groups are nevertheless interesting: for instance, the (-1)-st homotopy group of $\mathfrak{pic}(\mathrm{KU})^{\mathrm{h}C_2}$ computes the *relative Brauer group* of $(\mathrm{KO}, \mathrm{KU})$; see [GL21; Mor23b].

In particular, for $k = 0$, this spectral sequence converges to Pic(KO), allowing us to compute what we want, up to actually computing the spectral sequence.

Before we compute it, we recall what the homotopy groups of $\mathfrak{pic}(A)$ look like, when A is an $\mathbf{E}_\infty$ ring: we have

$$\pi_k\mathfrak{pic}(A) \cong \begin{cases} \mathrm{Pic}(A) & \text{if } k = 0, \\ (\pi_0 A)^\times & \text{if } k = 1, \\ \pi_{k-1}A & \text{if } k \geqslant 2. \end{cases}$$

This suggests that there might be a close connection to the HFPSS for $\pi_*\mathfrak{pic}(\mathrm{KO})$ and the HFPSS for $\pi_*\mathrm{KO}$ (shifted one to the right). This is indeed the case. Mathew and Stojanoska prove that

- differentials in the HFPSS for $\pi_*\mathfrak{pic}(\mathrm{KO})$ between entries lying in the range $k + s \geqslant 4$ correspond to differentials in the HFPSS for $\pi_*\mathrm{KO}$ (shifted one to the right); see [MS16, Comparison Tool 5.2.4];
- near the boundary of this range, there is a formula for differentials in the Picard HFPSS in terms of differentials in the HFPSS for $\pi_*\mathrm{KO}$; see [MS16, Theorem 6.1.1].

A picture of the Picard HFPSS is given in [MS16, §7, Figure 4]. What one finds in the end is that on the E_∞-page, there are four $\mathbf{Z}/2$'s appearing in the 0-stem. In particular this shows that $\#\mathrm{Pic}(\mathrm{KO}) = 8$. But we already know that Pic(KO) contains $\mathbf{Z}/8$ as a subgroup, so we conclude that

$$\mathrm{Pic}(\mathrm{KO}) \cong \mathbf{Z}/8,$$

being generated by $\Sigma\mathrm{KO}$.

3 The $\mathrm{K}(n)$-local Picard group

Next, we turn to the main topic of today, which is

$$\mathrm{Pic}(\mathrm{Sp}_{\mathrm{K}(n)}).$$

Unlike the Picard group of all spectra, this group is very complicated. At height 1 it is given by

$$\mathrm{Pic}(\mathrm{Sp}_{\mathrm{K}(1)}) \cong \begin{cases} \mathbf{Z}_2 \times \mathbf{Z}/4 \times \mathbf{Z}/2 & \text{if } p = 2, \\ \mathbf{Z}_p \times \mathbf{Z}/2(p-1) & \text{if } p \text{ is odd.} \end{cases}$$

Remark 3.1. If one likes the interpretation of the Picard group as elements on which we can index homotopy groups, this computation has the following fun implication. At odd primes p, the number $\frac{1}{2}$ is a p-adic integer, so one can talk about $\pi_{1/2}(X)$ when X is a $\mathrm{K}(1)$-local spectrum. (In fact, at every height, the $\mathrm{K}(n)$-local Picard group has a $\mathbf{Z}_p$ in it.)

The $\mathbf{Z}/2$ in the $p = 2$ case is special: it is an *exotic* Picard element. This means the following: Hopkins–Mahowald–Sadofsky define an algebraic approximation

$$\mathrm{Pic}(\mathrm{Sp}_{\mathrm{K}(n)}) \longrightarrow \mathrm{Pic}_n^{\mathrm{alg}},$$

and elements in the kernel κ_n of this map are called **exotic**, because "algebra cannot distinguish between them".

In general, we do not know if this map to the algebraic approximation is surjective. Even the algebraic approximation itself is very hard to compute. Our knowledge is almost completely concentrated in heights 2 and below, and computing the height 2 case took a lot of work by a lot of mathematicians. An overview of the current situation is given in the introduction of [Bob+24].

Contrast this with the extremely simple case of $\mathrm{Pic}(\mathrm{E}_n)$, which by a similar argument as in Example 1.5 is isomorphic to $\mathbf{Z}/2$ for all heights and all primes. We could hope to set up a mechanism of going from this easy case to the hard case, in a similar way as before. This is in fact possible, because there is an equivalence

$$\mathbf{S}_{\mathrm{K}(n)} \simeq \mathrm{E}_n^{\mathrm{h}\mathbf{G}},$$

where $\mathbf{G}$ denotes the Morava stabiliser group, and $\mathbf{E}_n$ denotes Morava E-theory. However, this formula is only true if one interprets the right-hand side as some sort of *continuous* fixed points: $\mathbf{G}$ carries a (profinite) topology and in some sense acts continuously on E_n. The fixed points should take this continuity into account.

This notion of continuity is subtle. For instance, it cannot be appropriately described as a functor $\mathrm{B}\mathbf{G} \to \mathrm{Sp}$, where $\mathrm{B}\mathbf{G}$ is the ∞-category underlying the topological category with one point and $\mathbf{G}$ as its endomorphisms. Devinatz and Hopkins [DH04] describe a way to make this continuity precise. We can rephrase their constructions in more modern terminology as saying that they turn Morava E-theory into an object of *condensed mathematics*.

3.1 Intermezzo: condensed mathematics

Definition 3.2. Let ProFin denote the category of profinite sets. This can be turned into a site by defining a cover to be a family of maps that contains a finite subfamily that is jointly surjective.

Definition 3.3. Let $\mathcal{C}$ be an ∞-category with limits. The ∞-category $\mathrm{Cond}(\mathcal{C})$ of **condensed objects** of $\mathcal{C}$ is the ∞-category of (hypercomplete) sheaves on ProFin with values in $\mathcal{C}$.

One should think of an object of $\mathrm{Cond}(\mathcal{C})$ as an object of $\mathcal{C}$ "equipped with a topology". If X is a condensed object and S a profinite set, then $X(S)$ should be thought of as the object of "maps from S into X". For example, $X(*)$ is the 'underlying object', and if $S = \mathbf{N} \cup \{ \infty \}$, then $X(S)$ should be interpreted as "convergent sequences in X with a chosen limit point".

Example 3.4. Let $\mathbf{G}$ be a profinite group (e.g., the Morava stabiliser group). Then $\mathbf{G}$ represents the functor $\mathrm{Cont}(-, \mathbf{G})$ of continuous maps from profinite sets into $\mathbf{G}$. This is a condensed group (because the site is subcanonical), so that $\mathbf{G}$ define an object of $\mathrm{Cond}(\mathrm{Grp})$. This condensed group is commonly denoted by $\underline{\mathbf{G}}$, or perhaps even simply by $\mathbf{G}$. ▲

Next, there is also a notion of an object with a continuous $\mathbf{G}$-action, if $\mathbf{G}$ is a profinite group.

Definition 3.5. Let $\mathrm{ProFin}_{\mathbf{G}}$ denote the category of profinite sets with a continuous (in the usual sense) $\mathbf{G}$-action. We turn this into a site in the same way as ProFin.

Definition 3.6. Write $\mathrm{Cond}_{\mathbf{G}}(\mathcal{C})$ for the ∞-category of (hypercomplete) sheaves on $\mathrm{ProFin}_{\mathbf{G}}$ with values in $\mathcal{C}$.

If X is now a condensed $\mathbf{G}$-object, then $X(\mathbf{G})$ should be thought of as the underlying object, while $X(*)$ should be thought of as the "continuous fixed points $X^{\mathrm{h}\mathbf{G}}$". In general, if $H \subseteq \mathbf{G}$ is a subgroup, then $X(\mathbf{G}/H)$ should be thought of as the continuous H-fixed points.

Remark 3.7. Alternatively, one can consider the ∞-category of $\mathbf{G}$-objects in $\mathrm{Cond}(\mathcal{C})$. This should be equivalent to the above description by some formal nonsense argument, but I did not check this in detail.

To motivate this a little more, consider the following similar example.

Example 3.8. Let G be a finite group, and let Fin_G be the category of finite G-sets. Then a (hypercomplete) sheaf of spectra on Fin_G should be[§] the same as a naive G-spectrum. Under this correspondence, $X(G)$ is the underlying spectrum, and $X(G/H)$ is the H-fixed points. ▲

3.2 Condensed E-theory

Going forward, let $\mathbf{G}$ denote the Morava stabiliser group. We can encode the continuity of the action of $\mathbf{G}$ on E_n by upgrading E_n to be a sheaf on $\mathrm{ProFin}_{\mathbf{G}}$. Up to some technicalities, this is essentially a reformulation of the constructions of Devinatz–Hopkins; cf. Remark 3.10.

Proposition 3.9 ([Mor23a], §2.3). *There exists a hypercomplete sheaf $\mathcal{E}$ of $\mathbf{E}_\infty$-rings on* $\mathrm{ProFin}_{\mathbf{G}}$*, satisfying*

(1) $\mathcal{E}(\mathbf{G}/e) \simeq \mathrm{E}_n$,

(2) $\mathcal{E}(\mathbf{G}/H) \simeq \mathrm{E}_n^{\mathrm{h}H}$ *if* $H \subseteq \mathbf{G}$ *is a finite subgroup,*

(3) $\mathcal{E}(\mathbf{G}/\mathbf{G}) \simeq \mathbf{S}_{\mathrm{K}(n)}$.

We describe part of the proof, without worrying about the more technical points.

Proof sketch. Devinatz and Hopkins produce a sheaf

$$\mathcal{E}^\delta \colon \mathrm{Fin}_{\mathbf{G}}^{\mathrm{op}} \longrightarrow \mathrm{Sp}_{\mathrm{K}(n)}.$$

We now form the left Kan extension in the $\mathrm{K}(n)$-local setting:

$$\begin{array}{ccc} \mathrm{Fin}_{\mathbf{G}}^{\mathrm{op}} & \xrightarrow{\mathcal{E}^\delta} & \mathrm{Sp}_{\mathrm{K}(n)}. \\ \big\downarrow & \nearrow & \\ \mathrm{ProFin}_{\mathbf{G}}^{\mathrm{op}} & & \end{array}$$

Postcomposing this left Kan extension with the inclusion $\mathrm{Sp}_{\mathrm{K}(n)} \hookrightarrow \mathrm{Sp}$ yields the desired functor $\mathcal{E}$. Item (3) then follows from [DH04]. ■

Remark 3.10. More concretely, if $H \subseteq \mathbf{G}$ is a closed subgroup, then $\mathcal{E}(\mathbf{G}/H)$ is given by

$$\mathcal{E}(\mathbf{G}/H) = \mathrm{L}_{\mathrm{K}(n)} \operatorname*{colim}_{U \supseteq H} \mathcal{E}^\delta(U),$$

§ I phrase it like this because I did not check this in detail.

where the colimit ranges over open subgroups U containing H such that $\mathbf{G}/U$ is finite. This is exactly how Devinatz–Hopkins define the continuous fixed points spectrum $\mathbf{E}_n^{\mathrm{h}H}$. Note that this is a $\mathrm{K}(n)$-local colimit, rather than a colimit in all spectra; this is why we took the left Kan extension in $\mathrm{Sp}_{\mathrm{K}(n)}$ rather than in Sp.

Remark 3.11. I follow Mor in denoting this sheaf by $\mathcal{E}$. If one wants to take the condensed mindset seriously, then an argument could be made for abusing notation and simply writing E_n instead. (In the same way that one might not distinguish between the notation for a topological space and its underlying set, but rather relying on context to make this distinction clear.) See Remark 5.2 as well.

The question arises: what plays the role of a 'continuous HFPSS'? This turns out to be a spectral sequence naturally associated to the site $\mathrm{ProFin}_{\mathbf{G}}$: the *descent spectral sequence*.

4 Descent spectral sequences

To make the construction more transparent, let us work in the general context of a site $\mathcal{T}$. (If one wishes to be extra fancy, one can set up this story in an arbitrary ∞-topos instead.) Those unfamiliar with sites can instead think of spectrum-valued sheaves on a topological[¶] space; this will be a helpful intuition anyway.

Let F be a sheaf of spectra on $\mathcal{T}$. Suppose we want to understand the homotopy groups of the sections of F at a fixed $X \in \mathcal{T}$, i.e., the homotopy groups of the spectrum $F(X)$. As a first approximation, we will consider the *homotopy sheaves* of F.

Definition 4.1. Consider the presheaf of abelian groups on $\mathcal{T}$ given by

$$\pi_n \circ F \colon \mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{Ab}, \quad Y \longmapsto \pi_n(F(Y)).$$

The sheafification of this presheaf is the ***n*-th homotopy sheaf** of F, and is denoted by $\pi_n F$.

Remark 4.2. The sheaf $\pi_k F$ can be wildly different from $\pi_k \circ F$. For instance, it can happen (in interesting, non-degenerate examples!) that $\pi_k F$ vanishes identically, while $\pi_k(F(X))$ is nonzero for many X. Roughly speaking, the reason for this is that when sheafifying $\pi_k \circ F$, we forget the homotopies. For example, if $x \in \pi_k(F(X))$ becomes zero after pulling it back to a cover $p \colon Y \to X$, then x becomes zero in $(\pi_k F)(X)$. However, the homotopy between $p^*(x)$ and 0 need not descend down to X, so x need not be zero in $\pi_k(F(X))$.

As such, $\pi_k F$ is much simpler than $\pi_k \circ F$. This is a feature, not a bug: as yet another instance of going from algebra to topology, there is a spectral sequence starting with the

[¶]Bear in mind that when considering the site associated to a topological space, the topological space should *not* be viewed in a homotopy-invariant way: the site 'sees' much more than just the underlying homotopy type of the topological space. In fact, for sober spaces (e.g., Hausdorff spaces) can be recovered from their site of opens (a.k.a. its *locale*).

homotopy sheaves, and ending with the homotopy groups we were interested in. This is the *descent spectral sequence* (DSS), which is of the form (for $X \in \mathcal{T}$ a fixed object)

$$\mathrm{E}_2^{k,s} \cong \mathrm{H}^s(X, (\pi_{k+s}F)(X)).$$

Here $\mathrm{H}^s(X, -)$ denotes sheaf cohomology, i.e., the s-th derived functor of evaluation at X. If X is the terminal object of $\mathcal{T}$, then these sheaf cohomology groups are also denoted by $\mathrm{H}^s(\mathcal{T}, -)$, and thought of as the $\mathcal{T}$-sheaf cohomology.

Remark 4.3. In reality, this spectral sequence is not one of abelian groups, but is one of sheaves of abelian groups on $\mathcal{T}$: its pages live in the abelian category $\mathrm{Sh}(\mathcal{T}; \mathrm{Ab})$ of sheaves of abelian groups on $\mathcal{T}$.

Convergence in general can be quite problematic, but in the applications later on, it will converge. I will however not give the proofs for this; in other words, I will simply not worry about convergence in this talk.

Remark 4.4. Note that the 0-th sheaf cohomology is simply evaluation at X, so that the $(\pi_k F)(X)$ appear on the bottom of the E_2-page.

Example 4.5. Let $\mathcal{T}$ be the site $\mathrm{Fin}_{\mathbf{G}}$ of finite $\mathbf{G}$-sets, for $\mathbf{G}$ a finite group, as in Example 3.8. The DSS for $\mathcal{T}$ can be identified with the HFPSS for the naive $\mathbf{G}$-spectrum. ▲

As such, we will think of the DSS for the site $\mathrm{ProFin}_{\mathbf{G}}$ as a version of the 'continuous HFPSS'. It is a spectral sequence of the form (where X is a condensed $\mathbf{G}$-spectrum)

$$\mathrm{E}_2^{k,s} \cong \mathrm{H}^s(\mathrm{ProFin}_{\mathbf{G}}, X) \implies \pi_k X(*).$$

5 The $\mathrm{K}(n)$-local Picard spectral sequence

We are now ready to start setting up the relevant spectral sequence for computing Picard groups. We mimic the approach from before, first setting up a 'linear variant', of which a shifted version will ultimately appear in the Picard spectral sequence.

Theorem 5.1 ([Mor23a], §2.3).

(1) *There is an isomorphism of condensed abelian groups (i.e., sheaves of abelian groups on* ProFin*)*

$$\pi_k \mathcal{E} \cong \mathrm{Cont}(-, \pi_k \mathrm{E}_n),$$

where on the right-hand side, we consider $\pi_k \mathrm{E}_n$ with its natural profinite topology.

(2) *The DSS for $\mathcal{E}(\mathbf{G}/\mathbf{G})$ is isomorphic to the* $\mathrm{K}(n)$*-local* E_n*-based Adams spectral sequence*

$$\mathrm{E}_2^{k,s} \cong \mathrm{H}^s_{\mathrm{cont}}(\mathbf{G}, \pi_{k+s}\mathrm{E}_n) \implies \pi_k \mathbf{S}_{\mathrm{K}(n)}.$$

Remark 5.2. The expression $\mathrm{Cont}(-, \pi_k \mathrm{E}_n)$ is the condensed abelian group represented by the profinite abelian group $\pi_k \mathrm{E}_n$. It is also commonly denoted by $\underline{\pi_k \mathrm{E}_n}$, in which case the isomorphism would read as

$$\pi_k \mathcal{E} \cong \underline{\pi_k \mathrm{E}_n},$$

further emphasising the idea that $\mathcal{E}$ can be thought of as E_n "with a topology". This fits nicely with the abuse of notation suggested in Remark 3.11.

Remark 5.3. The identification of the E_2-pages is a statement about continuous group cohomology (in the sense of topological groups) agreeing with sheaf cohomology for the site $\mathrm{ProFin}_{\mathbf{G}}$. This is an identification which one cannot expect to hold in general. In the general case, the latter notion is arguably the better one. For instance, if X is a general spectrum, then then we can define a sheaf $\mathcal{E} \hat{\otimes} X$ as the $\mathrm{K}(n)$-localisation of $\mathcal{E} \otimes X$ (the sheafification of level-wise tensoring with X). Its homotopy sheaves are a condensed version of completed Morava E-homology, usually denoted $\mathrm{E}_n^{\wedge}(X)$ or $\mathrm{E}_n^{\vee}(X)$. Writing $\mathcal{E}_k^{\wedge}(X)$ for $\pi_k(\mathcal{E} \hat{\otimes} X)$, the resulting DSS would be of the form

$$\mathrm{E}_2^{k,s} \cong \mathrm{H}^s(\mathrm{ProFin}_{\mathbf{G}}, \mathcal{E}_k^{\wedge}(X)) \implies \pi_k \, \mathrm{L}_{\mathrm{K}(n)} \, X.$$

One cannot expect the topological version of such a setup to work in as much generality.

The $\mathrm{K}(n)$-local E_n-based Adams spectral sequence is well studied, making it useful for our goal of computing the $\mathrm{K}(n)$-local Picard group.

To set up a spectral sequence for this Picard group, we need to lift the equivalence

$$\mathbf{S}_{\mathrm{K}(n)} \simeq \mathrm{E}_n^{\mathrm{h}\mathbf{G}},$$

with the right-hand side the continuous fixed points as before, to a statement of the form

$$\mathrm{Sp}_{\mathrm{K}(n)} \simeq \mathrm{Mod}_{\mathrm{E}_n}^{\mathrm{h}\mathbf{G}}.$$

In particular, we need to equip $\mathrm{Mod}_{\mathrm{E}_n}$ with the structure of a condensed $\mathbf{G}$-object in ∞-categories. This should come from the condensed $\mathbf{G}$-structure on E_n. Concretely, consider the composite

$$\mathrm{Mod}_{(-)}(\mathrm{Sp}_{\mathrm{K}(n)}) \circ \mathcal{E} \colon \mathrm{ProFin}_{\mathbf{G}}^{\mathrm{op}} \longrightarrow \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}), \quad X \longmapsto \mathrm{Mod}_{\mathcal{E}(X)}(\mathrm{Sp}_{\mathrm{K}(n)}).$$

Theorem 5.4 ([Mor23a], Theorem 3.1). *The functor* $\mathrm{Mod}_{(-)}(\mathrm{Sp}_{\mathrm{K}(n)}) \circ \mathcal{E}$ *is a hypercomplete sheaf of presentable ∞-categories.*

Using that $\mathfrak{pic}$ preserves small limits (or specifically, the formulation of Proposition 2.7), we deduce the following.

Corollary 5.5. *The functor*

$$\mathfrak{pic}(\mathcal{E}) \colon \mathrm{ProFin}_{\mathbf{G}}^{\mathrm{op}} \longrightarrow \mathrm{Sp}_{\geqslant 0}, \quad X \longmapsto \mathfrak{pic}(\mathrm{Mod}_{\mathcal{E}(X)}(\mathrm{Sp}_{\mathrm{K}(n)})$$

is a hypercomplete sheaf of connective spectra.

The DSS for $\mathfrak{pic}(\mathcal{E})$ is what we were after. It is of the form

$$\mathrm{E}_2^{k,s} = \mathrm{H}^s(\mathrm{ProFin}_{\mathbf{G}}, \pi_{k+s}\mathfrak{pic}(\mathcal{E})) \implies \pi_k \mathfrak{pic}(\mathrm{Sp}_{\mathrm{K}(n)}). \tag{5.6}$$

Again, we can compute this in terms of more familiar objects.

Proposition 5.7 ([Mor23a], §3.2). *We have isomorphisms*

$$\pi_t \mathfrak{pic}(\mathcal{E}) \cong \underline{\pi_t \mathfrak{pic}(\mathrm{E}_n)}$$

and

$$\mathrm{H}^s(\mathrm{ProFin}_{\mathbf{G}}, \pi_t \mathfrak{pic}(\mathcal{E})) \cong \mathrm{H}^s_{\mathrm{cont}}(\mathbf{G}, \pi_t \mathfrak{pic}(\mathrm{E}_n)).$$

The same techniques that were used to relate the Picard spectral sequence for a finite Galois extension to the HFPSS for the Galois extension carry over to this situation. (Brutally summarised, this is essentially because it is 'just' a sheafy version of the latter.) This computes the spectral sequence (5.6) in a range, and some addional work has to be done to compute the rest of the zero-stem. Mor carries this out in [Mor23a, §4]; I will very briefly summarise some of his results.

In the end, this allows one to recover the height 1 calculation. It also gives, at any height, a structural explanation of the exotic Picard elements. (Beware that I am ever so slightly simplifying the following result.)

Theorem 5.8. *The algebraic Picard group* $\mathrm{Pic}_n^{\mathrm{alg}}$ *is detected by elements in stem* 0 *and filtration* 1 *in the spectral sequence* (5.6).

In particular, this gives one a spectral-sequence way of thinking about the question of when the algebraic approximation map

$$\mathrm{Pic}(\mathrm{Sp}_{\mathrm{K}(n)}) \longrightarrow \mathrm{Pic}_n^{\mathrm{alg}}$$

is surjective: this happens if and only if the elements in stem 0 and filtration 1 do not support differentials. The exotic Picard group is represented by elements in stem 0 and filtration at least 2.

Example 5.9. At height $n = 1$, the 0-stem of the spectral sequence (5.6) is concentrated in filtration 0 at odd primes, while at the prime 2 it has elements in filtration 0, 1 and 3. The exotic $\mathbf{Z}/2$ from before is detected by this element in filtration 3, which does not support, nor is it hit by differentials. ▲

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